

Is the Gravitational Force Elementary?

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The gravitational force is assumed to be mediated by spin 2 gravitons which are composed of four spin 1/2 (sub-) fermions. The dynamics is governed by a nonlinear spinorfield equation with nonperturbative Pauli-Villars regularization and canonical quantization. On the quantum level the fermion dynamics is expressed by a functional equation for algebraic state functionals, and the “graviton” (functional) states are defined to be solutions of the diagonal part of this equation. The effective interaction between such graviton states and fermions is studied by means of a weak mapping between the original functional equation and an effective functional equation for gravitons and fermions. In the low energy limit this mapping produces for the effective fermion dynamics (apart from selfinteractions) a Dirac equation in anholonomic coordinates coupled to the anholonomic connections of a gravitational field, i.e. the usual phenomenological expression for a gravitational force acting on spin 1/2 fermions.

PACS 04.50 Unified field theories and other theories of gravitation.
PACS 04.60 Quantum theory of gravitation.

PACS 11.10 Field theory.
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PACS 12.35 Composite models of particles.

1. Introduction

The gravitational force acts indiscriminately on any kind of matter or energy. Thus it also affects the material or energetic standards of length and time, and their deformation leads to a violation of the postulates of Euclidean geometry. Therefore the gravitational force is closely linked with geometry. The first geometrical theory of gravitational force is due to Riemann [1]. The presently accepted geometrical standard theory of gravitation is due to Einstein [2] and Hilbert [3] who started from different premises but ended up with the same conclusions. With the discovery of the quantum formalism this geometrical theory of gravitation ran into difficulties, as the Hilbert-Einstein Lagrangian leads to a nonrenormalizable quantum field theory. Thus, by this purely formal reason, in the last decades numerous approaches were made to replace the Hilbert-Einstein Lagrangian by a Lagrangian of a renormalizable quantum field theory and to explain conventional gravity as a limiting case of such new theories. The most prominent representatives of such theories are supergravity, strings, superstrings, etc. Apart from the fact that none of these theories have ultimately succeeded, the most important prob-

lem is not the formal nonrenormalizability but rather the question: what does quantized geometry mean?

If geometrical gravitational theories are taken seriously, their extension into the quantum domain means that any microscopic process immediately changes the geometrical frame apart from the statistical fluctuations of the metric. This all penetrating flexibility of geometry is in sharp contrast to basic assumptions of quantum theory. In order to exist, quantum theory needs conservation laws providing the basis for quantum numbers. If by a totally flexible geometry all conservation laws are destroyed there are no quantum numbers and no quantum theory. On the other hand, quantum theories of microscopic phenomena are the best established theories that ever existed. Thus one has to conclude that geometrical theories of gravity are only compatible with quantum theory if they are referred to fixed frames of reference allowing the formation of conservation laws and quantum numbers. This does not mean total geometrical rigidity, because related with the material or energetic distributions one has to define “effective” metrics, but it means the primacy of quantum theory over flexible geometry.

This point of view was already emphasized by Edington [4], and thoroughly discussed by Dehnen, Hönl and Westpfahl [5]. Recently essential contributions were made by Borszeszkowski and Treder [6]

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and by Audretsch, Hehl and Lämmerzahl [7]. Apart from these basic discussions numerous proposals were made about gravity theories with fixed (unobservable) background metric and observable effective metric. The drawback of such approaches is in general their ad hoc character. By fixing the background metric, gravitation is formally described in terms of conventional (quantum) field theory and it is likely to expect a structural incorporation of gravity into the family of the other elementary forces. An attempt in this direction was made by Dehnen and Ghaboussi [8] who formulated gravitation as a gauge theory in terms of elementary particle physics.

An alternative systematic approach in this direction is fusion theory: All elementary forces including gravity are to be explained by the formation and interaction of elementary spin 1/2 fields. For relativistic particles this hypothesis was inaugurated by Eddington, Jordan and Heisenberg [9] who assumed the photon, i.e. the carrier of the electromagnetic forces, to be built up from a positive and a "negative" electron, i.e. spin 1/2 particles. Subsequently de Broglie [10] put forth the proposal that the photon is composed of a neutrino and an antineutrino and generalized this ansatz to include fusions of spin 1/2 particles to higher spin states. For instance, the graviton – the carrier of the gravitational force – was assumed to be a spin 2-state composed of four spin 1/2 fermions. The numerous investigations of such states were made by equations which described local fusions of spin 1/2 fields and led to free local composite one particle states. Such equations are for instance the Bargmann-Wigner equations. Recently there has been a renewed interest in the group theoretical analysis of graviton states resulting from Bargmann-Wigner equations. Rodriguez and Lorente [11] showed that such gravitons have properties which lead to vanishing curvature or trivial gravity. However, this difficulty can easily be removed if the graviton states are assumed to be multilocal. In addition, one needs nontrivial interactions. Both these requirements can be satisfied if one defines "graviton" states in the framework of a nonlinear spinorfield quantum theory, as was for instance proposed by Heisenberg [12]. The evaluation of such nonlinear spinorfield quantum theories depends on the ability to formulate a composite particle effective dynamics. An algebraic treatment of composite particle dynamics for quantum fields was recently developed [13]. In this paper we apply this theory to the problem of the interaction of "gravitons" with spin 1/2 fermions, i.e.

to the derivation of the gravitational force exerted on elementary spin 1/2 fermions. This problem is the most simple problem one can treat within the realm of spin fusion theory of gravity. A first outline of graviton selfinteraction was given in a preceding paper, where also additional references are given [14]. Compared with this previous paper, the discussion performed in this paper is based on a more detailed evaluation of the wave functions involved with this problem. The comparison with phenomenological theory is done by means of a lucid derivation of tetrad gravity and fermions given by Hehl [15].

2. Spinorfield Quantum Dynamics

The basic subcomponent fermions of our model are described by Dirac spinors which satisfy the following regularized nonlinear spinor-isospinor equation:

$$(i\gamma^\mu \partial_\mu - M)_{\alpha\beta}^{\text{reg}} \delta_{AB} \psi_{\beta B}(x) = g V_{\alpha\beta\gamma\delta}^{ABCD} \psi_{\beta B}(x) \bar{\psi}_{\gamma D}(x) \psi_{\delta D}(x) \quad (1)$$

with

$$\begin{aligned} (i\gamma^\mu \partial_\mu - M)_{\alpha\beta}^{\text{reg}} &:= ((i\gamma^\mu \partial_\mu - m_1)(i\gamma^\nu \partial_\nu - m_2)(i\gamma^\rho \partial_\rho - m_3))_{\alpha\beta}, \\ V_{\alpha\beta\gamma\delta}^{ABCD} &= \frac{1}{2} \sum_{h=1}^2 (v_{\alpha\beta}^h \delta_{AB} v_{\gamma\delta}^h \delta_{CD} - v_{\alpha\delta}^h \delta_{AD} v_{\gamma\beta}^h \delta_{CB}), \\ v_{\alpha\beta}^1 &:= \delta_{\alpha\beta}, \quad v_{\alpha\beta}^2 := i\gamma_{\alpha\beta}^5. \end{aligned}$$

By applying the decomposition theorem [16] to (1), we get the following equivalent set of nonlinear equations:

$$\begin{aligned} \delta_{AB}(i\gamma^\mu \partial_\mu - m_i)_{\alpha\beta} \varphi_{\beta Bi}(x) &= g \lambda_i \sum_{klm} V_{\alpha\beta\gamma\delta}^{ABCD} \varphi_{\beta Bk}(x) \bar{\varphi}_{\gamma Cl}(x) \varphi_{\delta Dm}(x) \quad (2) \end{aligned}$$

with

$$\begin{aligned} \varphi_{\alpha Ai}(x) &= \lambda_i [(i\gamma^\mu \partial_\mu - m_{i+1})(i\gamma^\nu \partial_\nu - m_{i+2})]_{\alpha\beta} \psi_{\beta A}(x), \\ \lambda_i &:= (m_i - m_{i+1})^{-1} (m_i - m_{i-1})^{-1}, \\ &\quad i = 1, 2, 3 \quad \text{cyclic}, \\ \psi_{\alpha Ai}(x) &= \sum_{i=1}^3 \varphi_{\alpha Ai}(x) \end{aligned}$$

which allow the application of the canonical quantization procedure. The λ_i fulfill regularization conditions

$$\sum_i \lambda_i = \sum_i \lambda_i m_i = 0. \quad (3)$$

If we use the charge conjugated spinor $\varphi_{\alpha Ai}^c$ instead of the adjoint spinor $\bar{\varphi}_{\alpha Ai}$ and furthermore introduce the

definitions

$$\phi_{\alpha\kappa i} := \{\varphi_{\alpha 1 i}, \varphi_{\alpha 2 i}, \varphi_{\alpha 1 i}^c, \varphi_{\alpha 2 i}^c\},$$

$$Z := (\alpha, \kappa, i),$$

$$D_{Z_1 Z_2}^\mu := i \gamma_{\alpha_1 \alpha_2}^\mu \delta_{\kappa_1 \kappa_2} \delta_{i_1 i_2},$$

$$m_{Z_1 Z_2} := m_{i_1} \delta_{\alpha_1 \alpha_2} \delta_{\kappa_1 \kappa_2} \delta_{i_1 i_2},$$

$$U_{Z_1 Z_2 Z_3 Z_4}^h := g \lambda_{i_1} v_{\alpha_1 \alpha_2}^h (v^k C)_{\alpha_3 \alpha_4} \delta_{\kappa_1 \kappa_2} \gamma_{\kappa_3 \kappa_4}^5,$$

we can combine (2) and its charge conjugated equations into one equation

$$(D_{Z_1 Z_2}^\mu \partial_\mu - m_{Z_1 Z_2}) \phi_{Z_2}(\mathbf{r}, t) = \sum_h U_{Z_1 \{Z_2 Z_3 Z_4\}_{\text{as}}}^h \phi_{Z_2}(\mathbf{r}, t) \phi_{Z_3}(\mathbf{r}, t) \phi_{Z_4}(\mathbf{r}, t), \quad (4)$$

and the canonical equal time anticommutator reads

$$\{\phi_{Z_1}(\mathbf{r}_1, t), \phi_{Z_2}(\mathbf{r}_2, t)\} = A_{Z_1 Z_2} \delta(\mathbf{r}_1 - \mathbf{r}_2),$$

$$A_{Z_1 Z_2} := \lambda_{i_1} \delta_{i_1 i_2} \gamma_{\kappa_1 \kappa_2}^5 (C \gamma^0)_{\alpha_1 \alpha_2}.$$

The quantum dynamics of the spinorfield is formulated by means of generating state functionals and corresponding functional equations. The quantum states $\{|a\rangle\}$ of the spinorfield are characterized by the set of matrix elements

$$a_n(\mathbf{r}_1, Z_1, \dots, \mathbf{r}_n, Z_n | a) = \langle 0 | \mathcal{A} \{\phi_{Z_1}(\mathbf{r}_1, t) \dots \phi_{Z_n}(\mathbf{r}_n, t)\} | a \rangle, \quad n=1, \dots, \infty$$

for equal times $t_1 = \dots = t_n = t$, where \mathcal{A} means antisymmetrization with respect to the set of indices $\{\mathbf{r}_1, Z_1\} \dots \{\mathbf{r}_n, Z_n\}$, and the corresponding generating functional states are defined by

$$|\mathcal{A}(j, a)\rangle = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int a_n(\mathbf{r}_1, Z_1 \dots \mathbf{r}_n, Z_n | a) \cdot j_{Z_1}(\mathbf{r}_1) \dots j_{Z_n}(\mathbf{r}_n) | 0 \rangle_F d^3 \mathbf{r}_1 \dots d^3 \mathbf{r}_n. \quad (6)$$

For these functional states corresponding functional equations can be derived. But we do not explicitly write down these equations because for the derivation of composite particle dynamics the functional states (6) are not the appropriate starting point.

In order to relate the original field theory of the spinorfield to composite particle dynamics it is necessary to remove vacuum expectation values from (6), as otherwise a composite particle interpretation of functional states would be impossible. For the case under consideration it is sufficient to remove only fermion propagators. This will be justified in the following sections. The removal of fermion propagators can be

performed by transition to “normal” ordered functional states with

$$|\mathcal{F}\rangle := Z_F[j] |\mathcal{A}\rangle, \quad (7)$$

$$Z_F[j] = \exp \left\{ \frac{1}{2} \int j_Z(\mathbf{r}) F(\mathbf{r}, \mathbf{r}') j_{Z'}(\mathbf{r}') d^3 \mathbf{r} d^3 \mathbf{r}' \right\}. \quad (8)$$

For brevity we introduce a collective index $I := (\mathbf{r}, Z)$ and write for $|\mathcal{F}\rangle$

$$|\mathcal{F}(j, a)\rangle := \sum_{n=0}^{\infty} \frac{i^n}{n!} \varphi_n(I_1 \dots I_n | a) j_{I_1} \dots j_{I_n} | 0 \rangle_F, \quad (9)$$

where the summation convention replaces integrations, etc. Then we get as a compact formulation of the spinorfield dynamics the functional energy equation

$$E_{a0} |\mathcal{F}(j, a)\rangle = K_{I_1 I_2} j_{I_1} \partial_{I_2} |\mathcal{F}(j, a)\rangle + \sum_h W_{I_1 I_2 I_3 I_4}^h \{ j_{I_1} \partial_{I_4} \partial_{I_3} \partial_{I_2} - 3 F_{I_4 I} j_{I_1} j_I \partial_{I_3} \partial_{I_2} + (3 F_{I_4 I} F_{I_3 I'} + \frac{1}{4} A_{I_4 I} A_{I_3 I'}) j_{I_1} j_I j_{I'} \partial_{I_2} - (F_{I_4 I} F_{I_3 I'} + \frac{1}{4} A_{I_4 I} A_{I_3 I'}) F_{I_2 I'} j_{I_1} j_I j_{I'} \} |\mathcal{F}(j, a)\rangle \quad (10)$$

with

$$E_{a0} := E_a - E_0.$$

$$K_{I_1 I_2} := i D_{Z_1 Z}^0 (D_{Z Z_2} \cdot \nabla_{\mathbf{r}_1} - m_{Z Z_2}) \delta(\mathbf{r}_1 - \mathbf{r}_2), \quad (11)$$

$$W_{I_1 I_2 I_3 I_4}^h := i D_{Z_1 Z}^0 U_{Z \{Z_2 Z_3 Z_4\}_{\text{as}}}^h \delta(\mathbf{r}_1 - \mathbf{r}_2) \delta(\mathbf{r}_1 - \mathbf{r}_3)$$

$$A_{I_1 I_2} := A_{Z_1 Z_2} \delta(\mathbf{r}_1 - \mathbf{r}_2) \cdot \delta(\mathbf{r}_1 - \mathbf{r}_4),$$

With respect to the definitions of functional states and the derivation of functional equations we refer to preceding papers.

3. Weak Mapping with Gravitons

In accordance with [14] we consider the appearance of the gravitational force as a composite particle effect. In this picture the gravitational force is exerted and mediated by spin-2 bosons which are assumed to be generated by four subfermion hard-core bound states of the spinorfield (1). As a consequence, the dynamical equations which characterize gravitational forces have to be derived as effective theories resulting from the spinorfield (1). The means for the derivation of effective theories is weak mapping.

Exact weak mapping theorems were derived for the two-fermion hard-core bound state dynamics [17] and for the combined two-fermion and three-fermion hard-core bound state dynamics [18]. These theorems lead

to rather complicated effective theories. If exchange effects between the bound states can be approximately neglected the effective dynamics becomes essentially simplified and can be derived by a short-cut functional calculation technique [19]. By this technique weak mapping was originally introduced, and one can treat any kind of composite particle configurations. Hence we apply it to gravitational boson mapping.

As we will see, the weak mapping with four (sub-) fermion bound states simultaneously enforces the introduction of dressed fermions instead of bare fermions. Hence in the simplest case we have at least to discuss a combined graviton-dressed fermion system. In this section we treat the formal aspects of its effective dynamics.

According to the short-cut calculation technique we define weak mapping by a transformation of the sources $\{j_I\}$ of the generating functional state (9). In full generality this transformation should deal with dressed particle states. The formal theory of dressed particle states was developed in [20] and [13]. They are assumed to be solutions of the complete functional equation (10) and thus contain an infinite number of polarization cloud terms induced by a hard-core wave function. To circumvent the difficulties connected with this infinite number of polarization cloud terms one is forced to consider only the lowest order in this expansion. This can be justified by estimates of the contributions of the polarization cloud terms to the total state norm and will be demonstrated for dressed fermion states in the following.

In the case under consideration the lowest order of the weak mapping transformation reads for gravitons

$$b_{4,k} := C_{4,k}^{I_1 I_2 I_3 I_4} j_{I_1} j_{I_2} j_{I_3} j_{I_4} \quad (12)$$

and for fermions

$$f_{1,k} := C_{1,k}^{I_1} j_{I_1} + C_{1,k}^{I_1 I_2 I_3} j_{I_1} j_{I_2} j_{I_3} \quad (13)$$

and leads to transformed source operators $\{b_{4,k}\}$ for hard-core gravitons and $\{f_{1,k}\}$ for first order dressed fermions.

Except for a center of mass part, which will be specified in Sect. 4, the set $\{C_{4,k}^{I_1 I_2 I_3 I_4}\}$ is assumed to be a set of solutions of the corresponding four-fermion hard-core equation which is defined as the diagonal part of (10), while the set $\{C_{1,k}^{I_1}, C_{1,k}^{I_1 I_2 I_3}\}$ is assumed to be a complete set of first order solutions of the full equation (10) in the spin 1/2 and fermion number 1 sector of the states (9).

Apart from bound state solutions the set of states $\{C_{4,k}^{I_1 I_2 I_3 I_4}\}$ also contains scattering solutions. Due to large fermion masses and the decoupling theorem the scattering solutions will be suppressed in the final evaluation of weak mapping results. But for general considerations we refer to the complete set $\{C_{4,k}^{I_1 I_2 I_3 I_4}\}$.

To carry out weak mapping, in addition to the primary sets $\{C_{4,k}^{I_1 I_2 I_3 I_4}\}$, $\{C_{1,k}^{I_1}, C_{1,k}^{I_1 I_2 I_3}\}$ their duals $\{R_{I_1 I_2 I_3 I_4}^{4,k}, \{R_{I_1}^{1,k}, R_{I_1 I_2 I_3}^{1,k}\}$ are needed. These dual sets are defined by orthonormality relations for dressed particle states [20] and lead in the lowest order approximation for graviton and fermion states to the relations

$$R_{I_1}^{1,k} C_{1,k'}^{I_1} + R_{I_1 I_2 I_3}^{1,k} C_{1,k'}^{I_1 I_2 I_3} = \delta_{1,k,1,k'} \quad (14)$$

and

$$R_{I_1 I_2 I_3 I_4}^{4,k} C_{4,k'}^{I_1 I_2 I_3 I_4} = \delta_{4,k,4,k'} \quad (15)$$

Furthermore, the inverse relations to (14) and (15) read

$$j_{I_1} = \sum_k R_{I_1}^{1,k} f_{1,k}, \quad (16)$$

$$j_{I_1} j_{I_2} j_{I_3} = \sum_k R_{I_1 I_2 I_3}^{1,k} f_{1,k}, \quad (17)$$

$$j_{I_1} j_{I_2} j_{I_3} j_{I_4} = \sum_k R_{I_1 I_2 I_3 I_4}^{4,k} b_{4,k}. \quad (18)$$

In specializing the general formulae [13] to (14)–(18) we have omitted all other kinds of particles which do not contribute to the graviton-fermion interaction. Furthermore we remark that in the present case we do not need the graviton duals $R^{4,k}$, i.e. the completeness of the graviton states $C^{4,k}$.

We assume that the generating functional state of the combined graviton-fermion effective theory is defined by

$$|\mathcal{G}(b, f, a)\rangle := \sum_{m,l} \frac{i^l}{l!} \frac{1}{m!} \quad (19)$$

$$\cdot \Theta(k_1 \dots k_m, h_1 \dots h_l | a) b_{4,k_1} \dots b_{4,k_m}$$

and that the transformation (12), (13) induces a rearrangement of the original generating functional state (9), i.e. (19) describes the same state as (9) but only formulated within another frame of reference. This can be expressed by the invariance relation

$$|\mathcal{F}(j, a)\rangle = |\mathcal{G}(b, f, a)\rangle. \quad (20)$$

If we write (10) in the compact form

$$\mathcal{H}(j, \partial) |\mathcal{F}(j, a)\rangle = E_{a0} |\mathcal{F}(f, a)\rangle, \quad (21)$$

then the transformation (12), (13) induces a transformation of the functional energy operator $\mathcal{H}(j, \partial)$ and (21) is transformed into

$$\hat{\mathcal{H}}\left(b, f, \frac{\delta}{\delta b}, \frac{\delta}{\delta f}\right)|\mathcal{G}(b, f, a)\rangle = E_{a0}|\mathcal{G}(b, f, a)\rangle. \quad (22)$$

The transformation of \mathcal{H} is generated by the functional chain rule and by (16), (17), (18), apart from higher order polarization cloud terms. This chain rule reads

$$\partial_I|\mathcal{F}(j, a)\rangle = \left[(\partial_I b_{4,k})\frac{\delta}{\delta b_{4,k}} + (\partial_I f_{1,k})\frac{\delta}{\delta f_{1,k}}\right]|\mathcal{G}(b, f, a)\rangle, \quad (23)$$

and repeated application yields $\partial_I \partial_I |\mathcal{F}(j, a)\rangle$ and $\partial_I \partial_I \partial_I |\mathcal{F}(j, a)\rangle$ in terms of boson and dressed

fermion derivatives acting on $|\mathcal{G}(b, f, a)\rangle$. In lowest order the functional derivatives of $b_{4,k}$ and $f_{1,k}$ are

$$\partial_I b_{4,k} = 4 C_{4,k}^{I I_2 I_3 I_4} j_{I_2} j_{I_3} j_{I_4}, \quad (24)$$

$$\partial_I f_{1,k} = C_{1,k}^I, \quad (25)$$

and by repeated application of ∂_I higher derivatives of $b_{4,k}$ and $f_{1,k}$ can be derived. In transforming $\mathcal{H}(j, \partial)$ by means of (23), (24), (25) and corresponding higher derivatives, we omit all monomials $j_{I_1} \dots j_{I_n}$ for $n > 4$, as these monomials are connected with higher order polarization terms which have only very small norm contributions.

For brevity we introduce the following symbols:

$$b_{4,k} \equiv b_k, \quad f_{1,k} \equiv f_k, \quad \frac{\delta}{\delta b_{4,k}} \equiv \partial_k^b, \quad \frac{\delta}{\delta f_{1,k}} \equiv \partial_k^f. \quad (26)$$

Using this notation we then obtain with (16)–(18)

$$\begin{aligned} \mathcal{H}(\partial, j)|\mathcal{F}(j, a)\rangle = & K_{I_1 I'} [4 C_{4,k}^{I' I_2 I_3 I_4} R_{I_1 I_2 I_3 I_4}^{4,k'} b_k \partial_k^b + C_{1,l}^{I'} R_{I_1}^{1,l'} f_l \partial_l^f] |\mathcal{G}(b, f, a)\rangle \\ & + \sum_h W_{I_1 I_2 I_3 I_4}^h \{ 36 C_{4,k}^{I_3 I_4 I' I''} C_{1,l}^{I_2} R_{I_1 I' I''}^{1,l'} f_l \partial_l^f \partial_k^b - 12 C_{4,k}^{I_4 I' I''} C_{1,l}^{I_3} C_{1,l'}^{I_2} R_{I_1 I' I''}^{4,k'} \partial_l^f \partial_l^f b_k \partial_k^b \\ & - C_{1,l}^{I_4} C_{1,l'}^{I_3} C_{1,l''}^{I_2} R_{I_1}^{1,j} f_j \partial_{l''}^f \partial_l^f \partial_l^f \\ & - 3 F_{I_4 I} [12 C_{4,k}^{I_2 I_3 I' I''} R_{I_1 I' I''}^{4,k'} b_k \partial_k^b + C_{1,l}^{I_2} C_{1,l'}^{I_3} R_{I_1}^{1,j} R_{I_1}^{1,j'} f_j f_{j'} \partial_l^f \partial_l^f] \\ & + (3 F_{I_4 I} F_{I_3 I'} + \frac{1}{4} A_{I_4 I} A_{I_3 I'}) R_{I_1}^{1,j} R_{I_1}^{1,j'} R_{I_1}^{1,j''} f_j f_{j'} f_{j''} C_{1,l}^{I_2} \partial_l^f \\ & - (F_{I_4 I} F_{I_3 I'} + \frac{1}{4} A_{I_4 I} A_{I_3 I'}) F_{I_2 I''} R_{I_1}^{1,j} R_{I_1}^{1,j'} R_{I_1}^{1,j''} R_{I_1}^{1,j'''} f_j f_{j'} f_{j''} f_{j'''} \} |\mathcal{G}(b, f, a)\rangle. \end{aligned} \quad (27)$$

We apply the equation

$$E_{a0}|\mathcal{G}(b, f, a)\rangle = i \left[b_k \frac{\partial}{\partial t} \partial_k^b + f_l \frac{\partial}{\partial t} \partial_l^f \right] |\mathcal{G}(b, f, a)\rangle \quad (28)$$

and go back to the original functional equations by collecting all terms linear in b_k or f_l , respectively, and by fulfilling the corresponding expressions separately. This gives the boson equation

$$\begin{aligned} -i \frac{\partial}{\partial t} \partial_k^b |\mathcal{G}\rangle = & K_{I_1 I'} 4 C_{4,k}^{I' I_2 I_3 I_4} R_{I_1 I_2 I_3 I_4}^{4,k} \partial_k^b |\mathcal{G}\rangle \\ & + \sum_h W_{I_1 I_2 I_3 I_4}^h \{ -12 C_{4,k}^{I_4 I' I''} C_{1,l}^{I_3} C_{1,l'}^{I_2} R_{I_1 I' I''}^{4,k} \partial_l^f \partial_l^f \partial_k^b - 36 F_{I_4 I} C_{4,k}^{I_2 I_3 I' I''} R_{I_1 I' I''}^{4,k} \partial_k^b \} |\mathcal{G}\rangle \end{aligned} \quad (29)$$

and the fermion equation

$$\begin{aligned} -i \frac{\partial}{\partial t} \partial_l^f |\mathcal{G}\rangle = & K_{I_1 I'} C_{1,l}^{I'} R_{I_1}^{1,l} \partial_l^f |\mathcal{G}\rangle \\ & + \sum_h W_{I_1 I_2 I_3 I_4}^h \{ 36 C_{4,k}^{I_3 I_4 I' I''} C_{1,l}^{I_2} R_{I_1 I' I''}^{1,l} \partial_l^f \partial_k^b - C_{1,j}^{I_4} C_{1,l'}^{I_3} C_{1,l''}^{I_2} R_{I_1}^{1,l} \partial_{l''}^f \partial_l^f \partial_j^f \\ & - 3 F_{I_4 I} C_{1,j}^{I_2} C_{1,j'}^{I_3} R_{I_1}^{1,l} R_{I_2}^{1,l'} f_{l'} \partial_j^f \partial_{j'}^f + (3 F_{I_4 I} F_{I_3 I'} + \frac{1}{4} A_{I_4 I} A_{I_3 I'}) R_{I_1}^{1,l} R_{I_1}^{1,j} R_{I_1}^{1,j'} C_{1,l'}^{I_2} f_{j'} f_{j''} \partial_l^f \\ & - (F_{I_4 I} F_{I_3 I'} + \frac{1}{4} A_{I_4 I} A_{I_3 I'}) F_{I_2 I''} R_{I_1}^{1,l} R_{I_1}^{1,j} R_{I_1}^{1,j'} R_{I_1}^{1,j''} f_j f_{j'} f_{j''} \} |\mathcal{G}\rangle. \end{aligned} \quad (30)$$

We are not interested in the selfinteractions of dressed fermions. Rather we are exclusively interested in the covariant coupling of gravitons to fermions. Thus we study only the linear parts of (30)

$$-i \frac{\partial}{\partial t} \partial_I^f |\mathcal{G}\rangle = K_{I_1 I'} C_{1, I'}^{I'} R_{I_1}^{1, I'} \partial_{I'}^f |\mathcal{G}\rangle + \sum_h W_{I_1 I_2 I_3 I_4}^h 36 C_{4, k}^{I_3 I_4 I' I''} C_{1, I'}^{I_2} R_{I_1 I' I''}^{1, I} \partial_{I'}^f \partial_k^b |\mathcal{G}\rangle \quad (31)$$

and evaluate them in the following sections.

4. Graviton States

To evaluate the graviton-fermion coupling term in (31), the graviton “wave-functions” $\{C_{4, k}^{I_1 I_2 I_3 I_4}\}$ and the dressed fermion “wavefunctions” $\{C_{1, I'}^{I_1}, C_{1, I}^{I_1 I_2 I_3}\}$ with their duals $\{R_{I_1}^{1, I'}, R_{I_1 I_2 I_3}^{1, I}\}$ are needed. With respect to the graviton states the calculations are easier because in contrast to fermion states their duals are not required. Hence we start with graviton state calculations.

Due to our suppression of all higher order polarization cloud terms for graviton states it is sufficient to take into account only their hard-core parts. These hard-core wavefunctions $\{C_{4, k}^{I_1 I_2 I_3 I_4}\}$ are solutions of the diagonal part of (10) in the four-fermion sector. The diagonal part of (10) reads

$$E_{a0}^d |\mathcal{F}(j, a)\rangle^d = [j_{I_1} K_{I_1 I_2} \partial_{I_2} - 3 W_{I_1 I_2 I_3 I_4} F_{I_4 K} j_{I_1} j_K \partial_{I_3} \partial_{I_2}] |\mathcal{F}(j, a)\rangle^d \quad (32)$$

with a corresponding “diagonal” \equiv hard-core functional state $|\mathcal{F}(j, a)\rangle^d$.

Due to the algebraic degrees of freedom which are contained in the symbolic indices I of (32), even for the low fermion numbers $n = 2, 3, 4$ (32) is rather complicated. However, these complications can be reduced if (32) is related to its covariant counterpart equation

$$\hat{K}_{I_1 I_2} \hat{\partial}_{I_2} |\hat{\mathcal{F}}(\hat{j}, a)\rangle^d = 3 \hat{W}_{I_1 I_2 I_3 I_4} \hat{F}_{I_4 K} \hat{j}_K \hat{\partial}_{I_3} \hat{\partial}_{I_2} |\hat{\mathcal{F}}(\hat{j}, a)\rangle^d \quad (33)$$

with

$$\begin{aligned} \hat{K}_{I_1 I_2} &:= (D_{Z_1 Z_2}^\mu \partial_{\mu 1} - m_{Z_1 Z_2}) \delta(x_1 - x_2), \\ \hat{F}_{I_1 I_2} &:= i \lambda_{i_1} \delta_{i_1 i_2} (\gamma_5)_{\kappa_1 \kappa_2} [(-i \gamma^q \partial_{q1} + m_{i_1}) C]_{\alpha_1 \alpha_2} \Delta(x_1 - x_2, m_{i_1}), \\ \hat{W}_{I_1 I_2 I_3 I_4} &:= \sum_h U_{Z_1(Z_2 Z_3 Z_4)_{as}}^h \delta(x_1 - x_2) \delta(x_1 - x_3) \delta(x_1 - x_4) \end{aligned} \quad (34)$$

and covariant source operators $\hat{j}_I \equiv \hat{j}_Z(x)$, $\hat{\partial}_I \equiv \hat{\partial}_Z(x)$.

The hard-core functional states for $n = 4$ are defined by

$$|\mathcal{F}(j, 4, a)\rangle^d := \frac{1}{4!} \varphi_{4, a}^{I_1 I_2 I_3 I_4} j_{I_1} j_{I_2} j_{I_3} j_{I_4} |0\rangle_F \quad (35)$$

or

$$|\hat{\mathcal{F}}(\hat{j}, 4, a)\rangle^d := \frac{1}{4!} \hat{\varphi}_{4, a}^{I_1 I_2 I_3 I_4} \hat{j}_{I_1} \hat{j}_{I_2} \hat{j}_{I_3} \hat{j}_{I_4} |\hat{0}\rangle_F, \quad (36)$$

respectively, where in (35) $I \equiv (Z, r)$, while in (36) $I \equiv (Z, x)$.

The relation between (33) and (34) and between (35) and (36) can be established by corresponding theorems which for $n = 2$ were given in [21]. Their generalization to $n = 4$ is straightforward, so we do not explicitly discuss it.

The explicit solution of (33) can be achieved by application of a reduction technique developed in [22].

If we consider the four-dimensional Fourier transform $\varphi_{4, a} \left(\begin{smallmatrix} q_1 & q_2 & q_3 & q_4 \\ Z_1 & Z_2 & Z_3 & Z_4 \end{smallmatrix} \right)$ of $\hat{\varphi}_{4, a}$ in (36) we can define the following contractions:

$$\varphi_4(q_1 | q_3, q_4) = \int d^4 \eta \varphi_4(\eta, q_1 - \eta, q_3, q_4), \quad (37)$$

$$\varphi_4(q_1, q_3 |) = \int d^4 \xi \varphi_4(q_1 | \xi, q_3 - \xi), \quad (38)$$

where for brevity we omitted the Z -indices. Then by means of this reduction technique, from (33) an integral equation for $\varphi_{4,a}(q_1, q_3|)$ can be derived, and knowing this function the full $\varphi_{4,a}(q_1 q_2 q_3 q_4)$ can be reconstructed. We do not prove the corresponding theorem but refer to [23]. Even this contracted integral equation is not exactly soluble.

To properly approximate the wavefunctions of (33) we only consider a submanifold of solutions. As we will investigate gravitational effects we are in particular interested in four-fermion spin-2 states which can be related to the tensor expressions of relativistic gravity theory.

The restriction to spin-2 states suggests to imagine these states as being composed of two spin-1 vector states, i.e., we consider the four-fermion spin-2 state as a bound state of two vector bosons. In order to utilize this idea for an approximate calculation of the four-fermion states we further assume that in spite of their fusion the internal wave functions of the vector bosons are approximately conserved. In this case the remaining task is the calculation of the wavefunction for the center of mass motion of both bosons, i.e., the solution of the integral equation (33) or its contracted version, can be reduced to the calculation of this wavefunction.

The wavefunctions of the single vector bosons are given in [24], and with $F_{\mu\nu} := A_\mu k_\nu$ the Fourier transform can be written as

$$\varphi_2(q_1, q_2)_{\alpha_1\alpha_2} = \frac{2i g_v}{(2\pi)^4} T_{\alpha_1\alpha_2}^j \delta(k - q_1 - q_2) \left\{ A_\nu u^\nu(q_1 - q_2)_{\alpha_1\alpha_2} + F_{\mu\nu} v^{\mu\nu}(q_1 - q_2)_{\alpha_1\alpha_2} \right\} \quad (39)$$

with

$$\begin{aligned} u^\nu(p)_{\alpha_1\alpha_2} &= -\lambda_{i_1} \lambda_{i_2} \left(\frac{p^2}{4} - m_{i_1} m_{i_2} \right) \left(\frac{p^2}{4} - m_{i_1}^2 \right)^{-1} \left(\frac{p^2}{4} - m_{i_2}^2 \right)^{-1} (\gamma^\nu C)_{\alpha_1\alpha_2} \\ &=: A_{i_1 i_2}(p) (\gamma^\nu C)_{\alpha_1\alpha_2}, \end{aligned} \quad (40)$$

$$\begin{aligned} v^{\mu\nu}(p)_{\alpha_1\alpha_2} &= \frac{i}{2} \lambda_{i_1} \lambda_{i_2} (m_{i_1} + m_{i_2}) \left(\frac{p^2}{4} - m_{i_1}^2 \right)^{-1} \left(\frac{p^2}{4} - m_{i_2}^2 \right)^{-1} (\Sigma^{\mu\nu} C)_{\alpha_1\alpha_2} \\ &=: D_{i_1 i_2}(p) (\Sigma^{\mu\nu} C)_{\alpha_1\alpha_2} \end{aligned} \quad (41)$$

in leading term approximation, i.e. with omission of internal k -dependence and internal angular momenta.

If two of these wavefunctions are combined to form a four-fermion bound state, we assume that the internal wavefunctions (40), (41) remain unchanged. Then we apply the following ansatz for the four-fermion spin-2 hard core states:

$$\begin{aligned} \varphi_{4,k}(q_1 q_2 q_3 q_4)_{Z_1 Z_2 Z_3 Z_4} &= N_4 g_v^2 T_{\alpha_1\alpha_2}^j T_{\alpha_3\alpha_4}^i \delta(k - q_1 - q_2 - q_3 - q_4) \\ &\quad \times \left\{ u^\nu(q_1 - q_2)_{\alpha_1\alpha_2} u^\mu(q_3 - q_4)_{\alpha_3\alpha_4} g_{\mu\nu} [(q_1 + q_2) - (q_3 + q_4)] \right. \\ &\quad + u^\nu(q_1 - q_2)_{\alpha_1\alpha_2} v^{\sigma e}(q_3 - q_4)_{\alpha_3\alpha_4} \Gamma_{\nu[\sigma e]} [(q_1 + q_2) - (q_3 + q_4)] \\ &\quad + v^{\nu\kappa}(q_1 - q_2)_{\alpha_1\alpha_2} u^\mu(q_3 - q_4)_{\alpha_3\alpha_4} \bar{\Gamma}_{[\nu\kappa]\mu} [(q_1 + q_2) - (q_3 + q_4)] \\ &\quad \left. + v^{\nu\kappa}(q_1 - q_2)_{\alpha_1\alpha_2} v^{\sigma e}(q_3 - q_4)_{\alpha_3\alpha_4} R_{[\nu\kappa][\sigma e]} [(q_1 + q_2) - (q_3 + q_4)] \right\} \end{aligned} \quad (42)$$

with $a \equiv (k, s = 2)$; for brevity we suppress $s = 2$ in the following. Substitution of (42) into the contracted integral equation leads to a set of integral equations for the calculation of the unknown boson-boson bonding tensor functions $g_{\mu\nu}(z)$, $\Gamma_{\nu[\sigma e]}(z)$, $\bar{\Gamma}_{[\sigma e]\nu}(z)$, $R_{[\nu\kappa][\sigma e]}(z)$. In deducing these integral equations we eliminate the spinor algebra by projection.

As for weak mapping we are mainly interested in this spinor algebra, we can use a further approximation for the calculation of the tensor functions. By applying the mean value theorem on the right-hand side of these

integral equations we obtain the following set of equations in the leading term approximation:

$$g_{\mu\nu}(p) = -\frac{2ig_\nu}{(2\pi)^4} \int d^4z f\left(-\frac{p}{2} - z, -z\right) g_{\mu\nu}(p) + \frac{ig_\nu}{4(2\pi)^4} \frac{A^{-1}}{(16)^2} \int d^4z A\left(\frac{p}{2} - z\right) A\left(\frac{p}{2} + z\right) Q_{\nu\mu}^{\mu'\nu'} \hat{g}_{\mu'\nu'}(0|k), \quad (43)$$

$$\Gamma_{[\sigma\varrho]\nu}(p) = \frac{ig_\nu}{(2\pi)^4} A^{-1} \frac{1}{2(16)^2} \int d^4z A\left(\frac{p}{2} - z\right) A\left(\frac{p}{2} + z\right) Q_{\nu\sigma\varrho}^{\nu'\sigma'\varrho'} \hat{\Gamma}_{[\nu'\sigma']\varrho'}(0|k), \quad (44)$$

$$\bar{\Gamma}_{[\sigma\varrho]\nu}(p) = -\frac{2ig_\nu}{(2\pi)^4} \int d^4z f\left(-\frac{p}{2} - z, -z\right) \bar{\Gamma}_{[\sigma\varrho]\nu}(p) + \frac{ig_\nu}{(2\pi)^4} \frac{A^{-1}}{2(16)^2} \int d^4z A\left(\frac{p}{2} - z\right) A\left(\frac{p}{2} + z\right) Q_{\nu\sigma\varrho}^{\nu'\sigma'\varrho'} \hat{\bar{\Gamma}}_{[\nu'\sigma']\varrho'}(0|k), \quad (45)$$

$$R_{[\mu\nu][\sigma\varrho]}(p) = \frac{ig_\nu}{(2\pi)^4} \frac{AD^{-2}}{4(16)^2} \int d^4z A\left(\frac{p}{2} - z\right) A\left(\frac{p}{2} + z\right) Q_{\mu\nu,\sigma\varrho}^{\mu'\nu'} \hat{g}_{\mu'\nu'}(0|k) \quad (46)$$

with

$$A(p) := \sum_{i_1 i_2} A_{i_1 i_2}(p); \quad A := \int d^4p A(2p),$$

$$D := \sum_{i_1 i_2} \int d^4p D_{i_1 i_2}(2p), \quad (47)$$

$$f(q, p) = \sum_{i_1 i_2} \lambda_{i_1} \lambda_{i_2} \frac{-pq + m_{i_1} m_{i_2}}{(p^2 - m_{i_1}^2)(q^2 - m_{i_2}^2)}, \quad (48)$$

and

$$Q_{\nu\mu}^{\nu'\mu'} := \eta_{\nu}^{\nu'} \eta_{\mu}^{\mu'} - \eta_{\nu\mu} \eta^{\mu'\nu'}, \quad (49)$$

$$Q_{\nu\sigma\varrho}^{\nu'\sigma'\varrho'} := \eta_{\sigma}^{\sigma'} \eta_{\varrho}^{\varrho'} \eta_{\nu}^{\nu'} \Big|_{\text{as}(\sigma, \varrho)}^{\text{as}(\sigma', \varrho')},$$

$$Q_{\mu\nu\sigma\varrho}^{\mu'\nu'\sigma'\varrho'} := \eta_{\nu}^{\nu'} \eta_{\sigma}^{\sigma'} \eta_{\mu\varrho}^{\mu'\varrho'} \Big|_{\text{as}(\mu\nu)}^{\text{as}(\mu'\nu')},$$

while the quantities (50)

$$\hat{X}(0|k) := \{\hat{g}_{\mu\nu}(0|k), \hat{\Gamma}_{[\sigma\varrho]}(0|k), \hat{\bar{\Gamma}}_{[\sigma\varrho]\nu}(0|k), \hat{R}_{\mu\nu\sigma\varrho}(0|k)\}$$

are the bonding tensors at the origin in four-dimensional coordinate space. Due to the four-dimensional translational invariance of (33), the four-momentum k is a good quantum number of the states (36), and therefore the set $\hat{X}(0|k)$ has to depend on k .

The bonding tensors $\hat{X}(0|k)$ can be considered as the center of mass amplitudes of the graviton with momentum k . If we resolve (43)–(46) with respect to $X(p|k)$ and substitute the resulting expressions into (42), then $\varphi_{4,a}^{I_1 I_2 I_3 I_4}$ is represented by explicitly given functions and the center of mass amplitudes.

It is one of the principles of weak mapping to consider the center of mass amplitudes as the new dynamical variables of the effective theory for the composite particles. Thus we do not determine the bonding tensors $\hat{X}(0|k)$ by their eigenvalue equations resulting from (33) or approximately from (43)–(46), but allow their unrestricted variability in order to fulfill the dynamical equations of the corresponding effective theory.

The fact that the center of mass amplitudes are considered as the dynamical fields of the effective theory enforces a discussion of their physical meaning. In the present case the $\hat{X}(0|k)$ arise from spinorfield products. Therefore it is obvious to assume that $\hat{X}(0|k)$ have to be identified with anholonomic quantities. For anholonomic coordinates the metrical tensor is $\eta_{\mu\nu}$. Hence, if in accordance with the background field formulation of Sect. 1 we consider $\hat{g}_{\mu\nu}(0|k)$ as the effective field correction to the background metric $\eta_{\mu\nu}$, in anholonomic coordinates $\hat{g}_{\mu\nu}(0|k)$ has to vanish, $\hat{g}_{\mu\nu}(0|k) \equiv 0$. This requirement is compatible with (43). But due to (46) it also enforces $R_{\mu\nu\sigma\varrho} \equiv 0$. The latter consequence does not mean that $R_{\mu\nu\sigma\varrho}$ vanishes at all, because $R_{\mu\nu\sigma\varrho}$ was only constructed in leading term approximation. Taking into account the contributions of orbital angular momenta, in (46) $R_{\mu\nu\sigma\varrho}$ would couple to $\Gamma_{[\sigma\varrho]}$ and $\bar{\Gamma}_{[\sigma\varrho]\nu}$ and thus would not vanish. However, compared with the leading terms, the orbital angular momentum terms are much smaller. Therefore, from the beginning we do not take into account these R -terms in the weak mapping procedure.

So, from all terms of $\hat{X}(0|k)$ we have to take into account only the $\Gamma_{[\sigma\varrho]}$ and $\bar{\Gamma}_{[\sigma\varrho]\nu}$ terms. But even these terms can still be reduced. So far we have done our calculation without antisymmetrization. If we antisymmetrize the ansatz (42) with the subsidiary conditions $g_{\mu\nu} = R_{\mu\nu\sigma\varrho} = 0$, the remaining terms for $\Gamma_{[\sigma\varrho]}$ and $\bar{\Gamma}_{[\sigma\varrho]\nu}$ are equal if $\hat{\Gamma}_{[\sigma\varrho]} = \hat{\bar{\Gamma}}_{[\sigma\varrho]\nu}$. Although there is a slight difference between (44) and (45), this difference does not alter the spin algebra. Hence for a first exploration we restrict ourselves to $\Gamma_{[\sigma\varrho]}$.

Under these assumptions we Fouriertransform (42) into coordinate space and obtain for equal times $t_1 =$

$t_2 = t_3 = t_4 = 0$ the function

$$\begin{aligned} \varphi_{4,k}(\mathbf{r}_1 \mathbf{r}_2 \mathbf{r}_3 \mathbf{r}_4)_{Z_1 Z_2 Z_3 Z_4} \\ = N_4 g_v^2 T_{\kappa_1 \kappa_2}^j T_{\kappa_3 \kappa_4}^i e^{-i \frac{k}{4} (\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4)} \\ \cdot \hat{u}^v \left(\frac{1}{2} (\mathbf{r}_1 - \mathbf{r}_2) \right)_{\alpha_1 \alpha_2} \hat{v}^{\sigma \varrho} \left(\frac{1}{2} (\mathbf{r}_3 - \mathbf{r}_4) \right)_{\alpha_3 \alpha_4} \\ \cdot \hat{f}_{v[\sigma \varrho]} \left(\frac{1}{4} (\mathbf{r}_1 + \mathbf{r}_2) - \frac{1}{4} (\mathbf{r}_3 + \mathbf{r}_4) \right). \end{aligned} \quad (51)$$

By means of (44), (47), (48), direct calculations yields

$$\hat{f}_{v[\sigma \varrho]}(\mathbf{r}) = v(\mathbf{r}) \hat{f}_{v[\sigma \varrho]}(0|\mathbf{k}) \quad (52)$$

with the regular function $v(\mathbf{r})$ which is given for $m\mathbf{r} \gg 1$ by

$$v(\mathbf{r}) = i z_0 g m^{-1} r e^{-4m\mathbf{r}}, \quad z_0 \in \mathbb{R}, \quad (53)$$

and

$$\hat{f}_{v[\sigma \varrho]}(0|\mathbf{k}) = Q_{v\sigma\varrho}^{v'\sigma'\varrho'} \hat{f}_{[v'\sigma']\varrho'}(0|\mathbf{k}). \quad (54)$$

Furthermore, according to [14] we assume $T^j \equiv T^i \equiv C$ and define

$$\begin{aligned} \hat{u}^v(\mathbf{r})_{\alpha_1 \alpha_2} &= \hat{A}_{i_1 i_2}(\mathbf{r}) (\gamma^v C)_{\alpha_1 \alpha_2}, \\ \hat{v}^{\sigma \varrho}(\mathbf{r})_{\alpha_3 \alpha_4} &= \hat{D}_{i_3 i_4}(\mathbf{r}) (\Sigma^{\sigma \varrho} C)_{\alpha_3 \alpha_4}. \end{aligned} \quad (55)$$

Then we obtain for (51)

$$\begin{aligned} \varphi_{4,k}(\mathbf{r}_1 \dots \mathbf{r}_4)_{Z_1 \dots Z_4} \\ = C_4 \left(\mathbf{r}_1 \dots \mathbf{r}_4 \left| \begin{matrix} \mathbf{k} \\ v, \sigma \varrho \end{matrix} \right| \right)_{Z_1 \dots Z_4} \hat{f}_{v[\sigma \varrho]}(0|\mathbf{k}) \end{aligned} \quad (56)$$

with

$$\begin{aligned} C_4 \left(\mathbf{r}_1 \dots \mathbf{r}_4 \left| \begin{matrix} \mathbf{k} \\ v, \sigma \varrho \end{matrix} \right| \right)_{Z_1 \dots Z_4} \\ = N_4 g_v^2 C_{\kappa_1 \kappa_2} C_{\kappa_3 \kappa_4} (\gamma^v C)_{\alpha_1 \alpha_2} (\Sigma^{\sigma \varrho} C)_{\alpha_3 \alpha_4} \\ \cdot e^{-i \frac{k}{4} (\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4)} \hat{A}_{i_1 i_2} \left(\frac{1}{2} (\mathbf{r}_1 - \mathbf{r}_2) \right) \\ \cdot \hat{D}_{i_3 i_4} \left(\frac{1}{2} (\mathbf{r}_3 - \mathbf{r}_4) \right) v \left(\frac{1}{4} (\mathbf{r}_1 + \mathbf{r}_2) - \frac{1}{4} (\mathbf{r}_3 + \mathbf{r}_4) \right). \end{aligned} \quad (57)$$

To see the role played by the dynamical center of mass amplitude (54) we consider a special case of (20), namely

$$\frac{1}{4!} \varphi_{4,k}(I_1 \dots I_4) j_{I_1} \dots j_{I_4} |0\rangle_F = \Theta(I|\mathbf{k})_{v,\sigma\varrho} b_{4,I}^{v,\sigma\varrho} |0\rangle \quad (58)$$

with

$$\Theta(I|\mathbf{k})_{v,\sigma\varrho} := \langle 0 | \Gamma_{v[\sigma\varrho]}^{\text{op}}(I) | \mathbf{k} \rangle.$$

On the other hand we obtain with (56) and (12)

$$\begin{aligned} \frac{1}{4!} \varphi_{4,k}(I_1 \dots I_4) j_{I_1} \dots j_{I_4} |0\rangle_F \\ = \frac{1}{4!} C_4 \left(I_1 \dots I_4 \left| \begin{matrix} \mathbf{k} \\ v, \sigma \varrho \end{matrix} \right| \right) \hat{f}_{v[\sigma \varrho]}(0|\mathbf{k}) j_{I_1} \dots j_{I_4} |0\rangle_F \\ = \frac{1}{4!} b_{4,k}^{v,\sigma\varrho} \hat{f}_{v[\sigma \varrho]}(0|\mathbf{k}). \end{aligned} \quad (59)$$

Comparison of (58) and (59) leads to

$$\Theta(I|\mathbf{k})_{v,\sigma\varrho} = \frac{1}{4!} \hat{f}_{v[\sigma \varrho]}(0|\mathbf{k}) \delta_{I\mathbf{k}}.$$

Hence the transformation matrices of (12) have to be identified with (57). This is of course only a heuristic interpretation. A rigorous deduction postulates (57) to be the transformation matrix of (12). Then the expansion coefficients of (19) are interpreted as matrix elements of corresponding phenomenological field operators which are assumed to be the quantized center of mass amplitudes.

Finally we want to emphasize that (57) has to be antisymmetrized. Due to the various subsymmetries and subantisymmetries the number of independent terms is reduced to six, as will be shown in section 6.

5. Dressed Fermion States

The calculation of the (first order) dressed fermion states can be performed along the same line as that for the boson functions. Again we consider the covariant multi-time functional equation and derive the energy equations as well as its solutions by a one-time limit. However, for dressed particles we need the full covariant functional equation. This reads

$$\hat{K}_{I_1 I_2} \hat{\delta}_{I_2} |\hat{\mathcal{F}}(\hat{j}, a)\rangle = \hat{W}_{I_1 I_2 I_3 I_4} \hat{d}_{I_4} \hat{d}_{I_3} \hat{d}_{I_2} |\hat{\mathcal{F}}(\hat{j}, a)\rangle \quad (60)$$

with

$$\hat{d}_I = \hat{\delta}_I - \hat{F}_{I\mathbf{r}} \hat{j}_{\mathbf{r}} \quad (61)$$

and definitions (34). By a one-time limit, (10) can be derived from (60).

In the lowest order of the covariant polarization cloud we represent the dressed fermion state by the functional

$$|\hat{\mathcal{F}}(\hat{j}, a)\rangle = \phi_{I_1} \hat{j}_{I_1} |0\rangle_F - \frac{i}{3!} \phi_{I_1 I_2 I_3} \hat{j}_{I_1} \hat{j}_{I_2} \hat{j}_{I_3} |0\rangle_F. \quad (62)$$

If we substitute (62) into (60) and project (60) into configuration space we obtain

$$(D_{I_1 I_2}^\mu \partial_\mu - m_{I_1 I_2}) \hat{\phi}_1(I_2) = g \hat{W}_{I_1 I_2 I_3 I_4} \hat{\phi}_3(I_2 I_3 I_4) \quad (63)$$

and

$$\begin{aligned} (D_{I_1 I_2}^\mu \partial_\mu - m_{I_1 I_2}) \hat{\phi}_3(N_1 N_2 I_2) \\ = 3g \hat{W}_{I_1 I_2 I_3 I_4} \{ [\hat{F}_{I_4 N_1} \hat{F}_{I_3 N_2} - \hat{F}_{I_4 N_2} \hat{F}_{I_3 N_1}] \hat{\phi}_1(I_2) \\ - [\hat{F}_{I_4 N_2} \hat{\phi}_3(N_1 I_2 I_3) + \hat{F}_{I_4 N_1} \hat{\phi}_3(N_2 I_2 I_3)] \}, \quad (64) \end{aligned}$$

where in (64) we suppressed the φ_5 -term due to our three source approximation in (62). In the same way, due to this approximation we also suppress all higher order equations for $\hat{\phi}_r$ with $r > 3$ which result from (60).

For a straightforward integration even these remaining equations (63) and (64) are too complicated. For a first inspection we apply an iteration procedure which we break off in the lowest order. We denote the lowest order functions by $\hat{\phi}_1^0$ and $\hat{\phi}_3^0$ and define the lowest order equations by

$$(D_{I_1 I_2}^\mu \partial_\mu - m_{I_1 I_2}) \hat{\phi}_1^0(I_2) = 0 \quad (65)$$

and

$$\begin{aligned} (D_{I_1 I_2}^\mu \partial_\mu - m_{I_1 I_2}) \hat{\phi}_3^0(N_1 N_2 I_2) \\ = 3g \hat{W}_{I_1 I_2 I_3 I_4} [\hat{F}_{I_4 N_1} \hat{F}_{I_3 N_2} - \hat{F}_{I_4 N_2} \hat{F}_{I_3 N_1}] \hat{\phi}_1^0(I_2). \quad (66) \end{aligned}$$

The other terms in (63) and (64) which do not appear in (65) and (66) are taken into account in the first iteration step.

Equations (65) and (66) can be directly integrated. While $\hat{\phi}_1^0$ is a Dirac superspinor, $\hat{\phi}_3^0$ is given by

$$\begin{aligned} \hat{\phi}_3^0(N_1 N_2 N_3) = \frac{g}{3!} \{ 3G_{N_3 I_1} \hat{W}_{I_1 I_2 I_3 I_4} [\hat{F}_{I_4 N_1} \hat{F}_{I_3 N_2} \\ - \hat{F}_{I_4 N_2} \hat{F}_{I_3 N_1}] \hat{\phi}_1^0(I_2) \}_{\text{as}(N_1 N_2 N_3)}. \quad (67) \end{aligned}$$

The Greensfunction $G_{NN'}$ in (67) reads

$$G_{NN'} = \frac{1}{(2\pi)^4} \delta_{\alpha\alpha'} \delta_{ii'} \int (q_\mu \gamma^\mu - m)_{\alpha\alpha'}^{-1} e^{iq(x-x')} dq. \quad (68)$$

In the limit of large m_i the integral (68) can be approximately represented by

$$\begin{aligned} \frac{1}{(2\pi)^4} \lim_{m \rightarrow \infty} \int (q_\mu \gamma^\mu - m)_{\alpha\alpha'}^{-1} e^{iq(x-x')} dq \\ \approx - \lim_{m \rightarrow \infty} \frac{1}{m} \delta(x-x') \delta_{\alpha\alpha'}. \quad (69) \end{aligned}$$

Due to the necessity of decoupling, the masses in (2) have to be very large. Hence we approximate (68) by

(69) and obtain

$$\lim_{m_i \rightarrow \infty} G_{NN'} \approx - \lim_{m_i \rightarrow \infty} \delta_{\alpha\alpha'} \delta_{ii'} \frac{1}{m_i} \delta(x-x') \delta_{\alpha\alpha'}. \quad (70)$$

If we substitute (70) into (67) we can perform the integration and afterwards proceed to equal times $t_1 = t_2 = t_3 = 0$. Furthermore, we can represent the propagators in the equal time limit by their leading terms [25]

$$F_{Z_1 Z_2}(\mathbf{r}_1 - \mathbf{r}_2) \quad (71)$$

$$\approx - (2\pi)^{-2} \lambda_{i_1} \delta_{i_1 i_2} \gamma_{\alpha_1 \alpha_2}^5 m_{i_1} \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|^2} C_{\alpha_1 \alpha_2}.$$

In general a subsequent auxiliary field summation produces a new order in leading terms. In the case of functions (71) we will, however, demonstrate that with respect to auxiliary field summation the leading term property of (71) is not destroyed. Hence this approximation is justified. If in addition the algebra is evaluated we obtain from (67) the expression

$$\begin{aligned} \varphi_3 = g N_3 \lambda_{i_1} \lambda_{i_2} \lambda_{i_3} m_{i_1} m_{i_2} m_{i_3}^{-1} \frac{1}{|\mathbf{r}_3 - \mathbf{r}_1|^2} \frac{1}{|\mathbf{r}_3 - \mathbf{r}_2|^2} \\ \cdot \{ [-\delta_{\alpha_3 \beta_2} C_{\alpha_1 \alpha_2} + \gamma_{\alpha_3 \beta_2}^5 (\gamma^5 C)_{\alpha_1 \alpha_2}] [\delta_{\alpha_3 \epsilon_2} \gamma_{\alpha_1 \alpha_2}^5] \\ + [\delta_{\alpha_1 \beta_2} C_{\alpha_3 \alpha_2} - \gamma_{\alpha_1 \beta_2}^5 (\gamma^5 C)_{\alpha_3 \alpha_2}] [\delta_{\alpha_1 \epsilon_2} \gamma_{\alpha_3 \alpha_2}^5] \\ + [-\delta_{\alpha_2 \beta_2} C_{\alpha_3 \alpha_1} + \gamma_{\alpha_2 \beta_2}^5 (\gamma^5 C)_{\alpha_3 \alpha_1}] [\gamma_{\alpha_3 \alpha_1}^5 \delta_{\alpha_2 \epsilon_2}] \} \\ \cdot \sum_{i'_2} \varphi_1 \begin{pmatrix} \mathbf{r}_3 & k \\ \beta_2 & h \\ \epsilon_2 & t \\ i'_2 & j \end{pmatrix}. \quad (72) \end{aligned}$$

In this form φ_3 is not properly antisymmetrized. If one tries to antisymmetrize (72), the last two terms of (72) lead to wrong results because the arrangement $\mathbf{r}_3, \alpha_3, \alpha_3, i_3$ for the arguments of φ_1 cannot be achieved. Hence we have to omit those terms. Then φ_3 reads

$$\varphi_3 = \{ [C_{\alpha_1 \alpha_2} \delta_{\alpha_3 \beta_2} - (\gamma^5 C)_{\alpha_1 \alpha_2} \gamma_{\alpha_3 \beta_2}^5] [\gamma_{\alpha_1 \alpha_2}^5 \delta_{\alpha_3 \epsilon_2}] \quad (73)$$

$$\cdot \lambda_{i_1} \lambda_{i_2} \lambda_{i_3} J \left(\begin{matrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 \\ i_1 & i_2 & i_3 \end{matrix} \right)_{\beta_2 \epsilon_2} \}_{\text{as}(\mathbf{r}_1 Z_1, \mathbf{r}_2 Z_2, \mathbf{r}_3 Z_3)}$$

with

$$J \left(\begin{matrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 \\ i_1 & i_2 & i_3 \end{matrix} \right)_{\beta_2 \epsilon_2} \quad (74)$$

$$:= -g N_3 m_{i_1} m_{i_2} m_{i_3}^{-1} \frac{1}{|\mathbf{r}_3 - \mathbf{r}_1|^2} \frac{1}{|\mathbf{r}_3 - \mathbf{r}_2|^2} \cdot \sum_{i'_2} \varphi_1 \begin{pmatrix} \mathbf{r}_3 & k \\ \beta_2 & h \\ \epsilon_2 & t \\ i'_2 & j \end{pmatrix}.$$

From (31) it follows that for the calculation of fermion-graviton coupling we need the dual state to (73). Dual states can be calculated by means of left-hand side solutions of (10). We immediately specialize to functional states (64) for the lowest order dressed one-fermion wavefunctions which in one-time limit read $\{C_{1,l}^{I_1}, C_{1,l}^{I_1 I_2 I_3}\}$. Let $\{L_{I_1}^{1,l}, L_{I_1 I_2 I_3}^{1,l}\}$ be a set of corresponding left-hand side wavefunctions, then we have the scalar product [27]

$$C_{1,l}^{I_2} L_{I_1}^{1,l'} + C_{1,l}^{I_1 I_2 I_3} L_{I_1 I_2 I_3}^{1,l'} = g^{1,l,1,l'}, \quad (75)$$

and by comparison with (14) we get

$$R_{I_1}^{1,l} = (g_{1,l,1,l'})^{-1} L_{I_1}^{1,l'}, \quad (76)$$

$$R_{I_1 I_2 I_3}^{1,l} = (g_{1,l,1,l'})^{-1} L_{I_1 I_2 I_3}^{1,l'}, \quad (77)$$

if $g^{1,l,1,l'}$ is diagonal.

We do not directly calculate $\{L_{I_1}^{1,l}, L_{I_1 I_2 I_3}^{1,l}\}$ but use the general formula for the scalar product [26]

$$g_{mk,m'k'} = \sum_{n,l} C_{m,k}^{*I_1 \dots I_n} G_{I_1 \dots I_n, I'_1 \dots I'_n}^{-1} C_{m',k'}^{I'_1 \dots I'_n} \quad (78)$$

in order to derive (76) and (77).

The scalar product (78) is bilinear in the amplitudes $C_{m,k}$. In evaluating a bilinear expression we have to observe that in general for limit processes $\lim(CGC) \neq (\lim C)G(\lim C)$ and that in addition the right-hand side may not exist at all. Hence we have to do our calculations ab initio, i.e. without taking into account limits like (70).

Specializing to the state $\{C_{1,k}^{I_1}, C_{1,k}^{I_1 I_2 I_3}\}$ we obtain by comparison of (78) with (75)

$$L_{I_1}^{1,k} = C_{1,k}^{*I_1} G_{I_1 I_1}^{-1} + C_{1,k}^{*I_1 I_2 I_3} G_{I_1 I_2 I_3, I_1}^{-1} \quad (79)$$

$$L_{I_1 I_2 I_3}^{1,k} = C_{1,k}^{*I_1} G_{I_1, I_1 I_2 I_3}^{-1} + C_{1,k}^{*I_1 I_2 I_3} G_{I_1 I_2 I_3, I_1 I_2 I_3}^{-1}. \quad (80)$$

The metric tensor G^{-1} can be calculated if one observes the iteration procedure (65)–(67). This procedure

implies that for dressed fermions a definite representation is introduced, which is defined by the solutions of (65). In this representation the metric tensor for normal-ordered basis states reads

$$G_{I_1, I_1} = \langle 0 | \phi_{I_1} \phi_{I_1}^+ | 0 \rangle, \quad (81)$$

$$G_{I_1, I_1 I_2 I_3} = \langle 0 | \phi_{I_1} \mathcal{N}(\phi_{I_1} \phi_{I_2} \phi_{I_3})^+ | 0 \rangle, \quad (82)$$

$$G_{I_1 I_2 I_3, I_1} = \langle 0 | \mathcal{N}(\phi_{I_1} \phi_{I_2} \phi_{I_3}) \phi_{I_1}^+ | 0 \rangle, \quad (83)$$

$$G_{I_1 I_2 I_3, I_1 I_2 I_3} = \langle 0 | \mathcal{N}(\phi_{I_1} \phi_{I_2} \phi_{I_3}) \mathcal{N}(\phi_{I_1} \phi_{I_2} \phi_{I_3})^+ | 0 \rangle, \quad (84)$$

and its evaluation yields [13]

$$G_{I_1 I_1} = \frac{1}{2} \lambda_i \delta_{I_1 I_1}, \quad (85)$$

$$G_{I_1 I_2 I_3, I_1 I_2 I_3} = \frac{1}{8} \lambda_{i_1} \lambda_{i_2} \lambda_{i_3} \delta_{I_1 I_1} \delta_{I_2 I_2} \delta_{I_3 I_3} |_{\text{as}(I_1 I_2 I_3)} \quad (86)$$

while the nondiagonal elements (82), (83) vanish. Thus, with (79), (80), formulae (76), (77) go over into

$$R_{I_1}^{1,l} = (g_{1,l,1,l'})^{-1} 2 \lambda_{i_1}^{-1} C_{1,l'}^{*I_1}, \quad (87)$$

$$R_{I_1 I_2 I_3}^{1,l} = 8 (g_{1,l,1,l'})^{-1} \lambda_{i_1}^{-1} \lambda_{i_2}^{-1} \lambda_{i_3}^{-1} C_{1,l'}^{*I_1 I_2 I_3}. \quad (88)$$

To derive $(g_{1,l,1,l'})^{-1}$ we substitute the explicit expression (74) into (78), which in this order reads

$$g_{1,k,1,k'} = C_{1,k}^{*I_1} G_{I_1 I_1}^{-1} C_{1,k'}^{I'_1} + C_{1,k}^{*I_1 I_2 I_3} G_{I_1 I_2 I_3, I_1 I_2 I_3}^{-1} C_{1,k'}^{I'_1 I'_2 I'_3}. \quad (89)$$

Direct evaluation yields

$$\begin{aligned} g_{1,l,1,l'} &= C_{1,l}^{*I_1} 2 \lambda_{i_1}^{-1} \delta_{I_1 I_1} C_{1,l'}^{I'_1} \left(1 + \frac{g^2}{m^4} \right) \\ &= 2 \lambda_{i_1}^{-1} \delta_{I_1 I_1} \left(1 + \frac{g^2}{m^4} \right), \end{aligned} \quad (90)$$

The contribution of the polarization cloud to the one-particle norm is therefore of order g^2/m^2 .

Finally it should be remarked that, in contrast to Sect. 4, in φ_3 no center of mass amplitude is used. Hence $\varphi_3 \equiv C_3$ directly holds.

6. Fermion-Graviton Coupling

The general expression for fermion-graviton coupling, i.e. anholonomic spinor connections is given by the last term of (31). By means of the wavefunctions of Sects. 4 and 5 we can evaluate this term in detail. In particular we use the graviton wavefunctions (57), and for the dual of the dressed fermion wavefunction the combination of (73), (74) and (88). Both of them are to be completely antisymmetrized. We write the mapping term in the

form

$$\begin{aligned}
 & W_{I(L_1 L_2 L_3)} C_K^{L_1 L_2 I_3 I_4} R_{II_3 I_4}^{1, k} C_{1, k'}^{L_3} \partial_{1, k'}^f \partial_K^b \\
 & = \int C \left(\begin{array}{cccc|c} r_1 & r_2 & r_3 & r_4 & K \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & A \\ \varkappa_1 & \varkappa_2 & \varkappa_3 & \varkappa_4 & \\ i_1 & i_2 & i_3 & i_4 & \end{array} \right) R \left(\begin{array}{ccc|c} r & r_3 & r_4 & k \\ \beta & \alpha_3 & \alpha_4 & h \\ \varepsilon & \varkappa_3 & \varkappa_4 & t \\ i & i_3 & i_4 & j \end{array} \right) C \left(\begin{array}{c|c} r' & k' \\ \varkappa' & h' \\ i' & t' \\ j' & \end{array} \right) \partial^f \left(\begin{array}{c} k' \\ h' \\ t' \\ j' \end{array} \right) \partial_{K, A}^b \\
 & \cdot i^2 g \lambda_i B^{i_1 i_2 i'} \gamma_{\beta \alpha}^0 V \left(\begin{array}{cccc} \alpha & \alpha_1 & \alpha_2 & \alpha' \\ \varepsilon & \varkappa_1 & \varkappa_2 & \varkappa' \end{array} \right) \delta(r-r_1) \delta(r-r_2) \delta(r-r') \\
 & \cdot d^3 r d^3 r' d^3 r_1 d^3 r_2 d^3 r_3 d^3 r_4 d^3 K d^3 k'.
 \end{aligned} \tag{91}$$

Substitution of the antisymmetrized wavefunctions in (91) yields the expression

$$(91) \equiv \sum_{\substack{i_1 i_2 i_3 i_4 \\ i i'}} \int d^3 r d^3 r_3 d^3 r_4 d^3 K d^3 k' e^{iK(2r+r_3+r_4)1/4} \tag{i}$$

$$\begin{aligned}
 & \frac{N_4 g_v^2}{3!} \{ C_{\varkappa_1 \varkappa_2} C_{\varkappa_3 \varkappa_4} (\gamma^\mu C)_{\alpha_1 \alpha_2} (\Sigma^{\sigma \lambda} C)_{\alpha_3 \alpha_4} A_{i_1 i_2}(0) D_{i_3 i_4}(\tfrac{1}{2}(r_3 - r_4)) v(\tfrac{1}{4}(2r - r_3 - r_4)) \cdot 1 \\
 & + C_{\varkappa_3 \varkappa_1} C_{\varkappa_2 \varkappa_4} (\gamma^\mu C)_{\alpha_3 \alpha_1} (\Sigma^{\sigma \lambda} C)_{\alpha_2 \alpha_4} A_{i_3 i_1}(\tfrac{1}{2}(r - r_3)) D_{i_2 i_4}(\tfrac{1}{2}(r - r_4)) v(\tfrac{1}{4}(r_3 - r_4)) \cdot 2 \\
 & + C_{\varkappa_1 \varkappa_4} C_{\varkappa_2 \varkappa_3} (\gamma^\mu C)_{\alpha_1 \alpha_4} (\Sigma^{\sigma \lambda} C)_{\alpha_2 \alpha_3} A_{i_1 i_4}(\tfrac{1}{2}(r - r_4)) D_{i_2 i_3}(\tfrac{1}{2}(r - r_3)) v(\tfrac{1}{4}(r_4 - r_3)) \cdot 3 \\
 & + C_{\varkappa_3 \varkappa_4} C_{\varkappa_1 \varkappa_2} (\gamma^\mu C)_{\alpha_3 \alpha_4} (\Sigma^{\sigma \lambda} C)_{\alpha_1 \alpha_2} A_{i_3 i_4}(\tfrac{1}{2}(r_3 - r_4)) D_{i_1 i_2}(0) v(\tfrac{1}{4}(r_3 + r_4 - 2r)) \cdot 4 \\
 & + C_{\varkappa_3 \varkappa_1} C_{\varkappa_2 \varkappa_4} (\gamma^\mu C)_{\alpha_2 \alpha_4} (\Sigma^{\sigma \lambda} C)_{\alpha_3 \alpha_1} A_{i_2 i_4}(\tfrac{1}{2}(r - r_4)) D_{i_1 i_3}(\tfrac{1}{2}(r - r_3)) v(\tfrac{1}{4}(r_4 - r_3)) \cdot 5 \\
 & + C_{\varkappa_2 \varkappa_3} C_{\varkappa_1 \varkappa_4} (\gamma^\mu C)_{\alpha_2 \alpha_3} (\Sigma^{\sigma \lambda} C)_{\alpha_1 \alpha_4} A_{i_2 i_3}(\tfrac{1}{2}(r - r_3)) D_{i_1 i_4}(\tfrac{1}{2}(r - r_4)) v(\tfrac{1}{4}(r_3 - r_4)) \} \cdot 6
 \end{aligned} \tag{i}$$

$$\begin{aligned}
 & \frac{1}{3!} \sum_h g \lambda_i \gamma_{\beta \alpha}^0 \{ v_{\alpha \alpha_1}^h (v^h C)_{\alpha_2 \alpha'} \delta_{\varepsilon \varkappa_1} \gamma_{\varkappa_2 \varkappa'}^5 \cdot 1 \\
 & - v_{\alpha \alpha_2}^h (v^h C)_{\alpha_1 \alpha'} \delta_{\varepsilon \varkappa_2} \gamma_{\varkappa_1 \varkappa'}^5 \cdot 2 \\
 & + v_{\alpha \alpha'}^h (v^h C)_{\alpha_1 \alpha_2} \delta_{\varepsilon \varkappa'} \gamma_{\varkappa_1 \varkappa_2}^5 \} \cdot 3
 \end{aligned} \tag{j}$$

$$\begin{aligned}
 & (-1) \lambda_j \frac{4}{3!} \sum_{h'} \left\{ \gamma_{\varepsilon \varkappa_3}^5 \delta_{\varkappa_4 \varrho} (v^{h'} C)_{\beta \alpha_3} (v^{h'})_{\delta \alpha_4} \left[J^* \left(\begin{array}{ccc} r & r_3 & r_4 \\ i & i_3 & i_4 \end{array} \right)_{\delta \varrho} + J^* \left(\begin{array}{ccc} r_3 & r & r_4 \\ i_3 & i & i_4 \end{array} \right)_{\delta \varrho} \right] \cdot 1 \right. \\
 & + \gamma_{\varkappa_3 \varkappa_4}^5 \delta_{\varepsilon \varrho} (v^{h'} C)_{\alpha_3 \alpha_4} (v^{h'})_{\delta \beta} \left[J^* \left(\begin{array}{ccc} r_3 & r_4 & r \\ i_3 & i_4 & i \end{array} \right)_{\delta \varrho} + J^* \left(\begin{array}{ccc} r_4 & r_3 & r \\ i_4 & i_3 & i \end{array} \right)_{\delta \varrho} \right] \cdot 2 \\
 & \left. + \gamma_{\varkappa_4 \varepsilon}^5 \delta_{\varkappa_3 \varrho} (v^{h'} C)_{\alpha_4 \beta} (v^{h'})_{\delta \alpha_3} \left[J^* \left(\begin{array}{ccc} r_4 & r & r_3 \\ i_4 & i & i_3 \end{array} \right)_{\delta \varrho} + J^* \left(\begin{array}{ccc} r & r_4 & r_3 \\ i & i_4 & i_3 \end{array} \right)_{\delta \varrho} \right] \right\} \cdot 3 \\
 & \cdot C \left(\begin{array}{c|c} r & k' \\ \alpha' & h' \\ \varkappa' & t' \\ i' & j' \end{array} \right) \partial^f \left(\begin{array}{c} k' \\ h' \\ t' \\ j' \end{array} \right) \partial_{K, \mu, \sigma \lambda}^b,
 \end{aligned} \tag{k}$$

where we have to observe that the quantum numbers of $R_{II3I4}^{1,k}$ are contained in definition (74). In particular λ_j^{-1} in (92) is referred to these quantum numbers.

The terms in (92) can be characterized by the combinations (i, j, k) , $1 \leq i \leq 6$, $1 \leq j \leq 3$, $1 \leq k \leq 3$. Due to the special form of (74) the sum over λ_i in $k = 1, 3$ yields the expression $\sum_i m_i \lambda_i = 0$. Hence $(i, j, 1) = 0$, $(i, j, 3) = 0 \forall i, j$. Thus we have to calculate only the terms $(i, j, 2) \forall i, j$. By exact evaluation we get for the algebraic parts $(2, 2, 2) = (3, 2, 2) = (5, 1, 2) = (6, 1, 2) = -4 \delta_{\kappa' \varrho} (\gamma^0 \Sigma^{\sigma \lambda} \gamma^\mu)_{\delta \alpha'}$, while all other terms $(i, j, 2)$ with $i, j \neq 2, 2, 3, 2, 5, 1, 6, 1$ vanish. Inserting these results into (92) and observing the symmetry properties of the coefficient functions we obtain

$$(91) \equiv N g_v^2 g \sum_{i i_1 i_2 i_3 i_4 i'} \int d^3 r d^3 r_3 d^3 r_4 d^3 K d^3 k' e^{i \mathbf{K} (2 \mathbf{r} + \mathbf{r}_3 + \mathbf{r}_4)} \frac{1}{4} \\ \times \lambda_i \lambda_j A_{i_3 i_1} \left(\frac{1}{2} (\mathbf{r} - \mathbf{r}_3) \right) D_{i_2 i_4} \left(\frac{1}{2} (\mathbf{r} - \mathbf{r}_4) \right) v \left(\frac{1}{2} (\mathbf{r}_3 - \mathbf{r}_4) \right) \left[J^* \left(\begin{matrix} \mathbf{r}_3 & \mathbf{r}_4 & \mathbf{r} \\ i_3 & i_4 & i \end{matrix} \right)_{\delta \varrho} + J^* \left(\begin{matrix} \mathbf{r}_4 & \mathbf{r}_3 & \mathbf{r} \\ i_4 & i_3 & i \end{matrix} \right)_{\delta \varrho} \right] \\ \times C \left(\begin{matrix} \mathbf{r} & \mathbf{k}' \\ \alpha' & h' \\ \kappa' & t' \\ i' & j' \end{matrix} \right) \delta_{\kappa' \varrho} (\gamma^0 \Sigma^{\sigma \lambda} \gamma^\mu)_{\delta \alpha'} \partial^f \left(\begin{matrix} \mathbf{k}' \\ h' \\ t' \\ j' \end{matrix} \right) \partial_{\mu, \sigma \lambda}^b (\mathbf{K}). \quad (93)$$

To evaluate (93) we define

$$J \left(\begin{matrix} \mathbf{r}_3 & \mathbf{r}_4 & \mathbf{r} \\ i_3 & i_4 & i \end{matrix} \right)_{\delta \varrho} = m_{i_3} m_{i_4} m_i^{-1} \hat{J}(\mathbf{r}_3 \mathbf{r}_4 \mathbf{r})_{\delta \varrho} \quad (94)$$

and substitute it into (93). Then we have to calculate the expressions $\sum_{i_3 i_1} A_{i_1 i_3} m_{i_3}$ and $\sum_{i_2 i_4} D_{i_2 i_4} m_{i_4}$. This can be done by setting $m_i = m + \Delta_i$ and by assuming Δ_i very small, $\Delta_i \ll m$ in comparison with m . In the one-time limit the leading terms of a power series expansion of $A_{i_1 i_3}$ and $D_{i_2 i_4}$ in Δ_i can then be calculated and yield together with the regularization conditions (3)

$$\sum_{i_1 i_3} A_{i_1 i_3}(\mathbf{r}) m_{i_3} = i c_1 m^3 \int e^{-i \mathbf{p} \cdot \mathbf{r}} (\mathbf{p}^2 + m^2)^{-\frac{7}{2}} d^3 p, \quad (95)$$

$$\sum_{i_2 i_4} D_{i_2 i_4}(\mathbf{r}) m_{i_4} = c_2 m \int e^{-i \mathbf{p} \cdot \mathbf{r}} (\mathbf{p}^2 + m^2)^{-\frac{5}{2}} d^3 p. \quad (96)$$

These Fouriertransforms can be directly calculated. They read

$$\sum_{i_1 i_3} A_{i_1 i_3}(\mathbf{r}) m_{i_3} = i c'_1 m^{\frac{1}{2}} r^{\frac{3}{2}} e^{-m r}, \quad (97)$$

$$\sum_{i_2 i_4} D_{i_2 i_4}(\mathbf{r}) m_{i_4} = c'_2 m^{-\frac{3}{2}} r^{\frac{1}{2}} e^{-m r}, \quad (98)$$

for large $m r \gg 1$ with real constants c'_1, c'_2 . For small r , in particular at $r = 0$, the integrals (95) and (96) are finite. Hence the functions (97), (98) are suitable testfunctions which are regular at the origin and have a rapid decrease at infinity. Therefore the integrals in (93) are finite. For explicit calculation we represent the one-particle states by

$$C \left(\begin{matrix} \mathbf{r} & \mathbf{k}' \\ \alpha' & h' \\ \kappa' & t' \\ i' & j' \end{matrix} \right) = e^{-i \mathbf{k}' \cdot \mathbf{r}} \delta_{i' j'} \chi \left(\begin{matrix} \mathbf{k}' \\ h' \\ t' \\ j' \end{matrix} \right)_{\alpha' \kappa'}, \quad \varphi_1^* \left(\begin{matrix} \mathbf{r} & \mathbf{k} \\ \delta & h \\ \varrho & t \\ i'_2 & j \end{matrix} \right) = e^{i \mathbf{k} \cdot \mathbf{r}} \delta_{i'_2 j} \chi^* \left(\begin{matrix} \mathbf{k} \\ h \\ t \\ j \end{matrix} \right)_{\delta \varrho}. \quad (99)$$

Then the integral in (93) is given by

$$\begin{aligned}
 I = N' \lambda_j g_v^2 g^2 \int \left[\sum_{i_1 i_3} A_{i_1 i_3} \left(\frac{1}{2} (\mathbf{r} - \mathbf{r}_3) \right) m_{i_3} \right] \left[\sum_{i_2 i_4} D_{i_2 i_4} \left(\frac{1}{2} (\mathbf{r} - \mathbf{r}_4) \right) m_{i_4} \right] \\
 \times v \left(\frac{1}{4} (\mathbf{r}_3 - \mathbf{r}_4) \right) \left(\sum_i \lambda_i m_i^{-1} \right) \frac{1}{|\mathbf{r} - \mathbf{r}_3|^2} \frac{1}{|\mathbf{r} - \mathbf{r}_4|^2} e^{i\mathbf{K}(2\mathbf{r} + \mathbf{r}_3 + \mathbf{r}_4) \cdot \frac{1}{4}} e^{i\mathbf{K}\mathbf{r}} e^{-i\mathbf{K}'\mathbf{r}} \\
 \times \chi \begin{pmatrix} \mathbf{k}' \\ h' \\ t' \\ j' \end{pmatrix}_{\alpha' \alpha'} \chi^* \begin{pmatrix} \mathbf{k} \\ h \\ t \\ j \end{pmatrix}_{\delta' \delta} d^3 r d^3 r_3 d^3 r_4 \partial^f \begin{pmatrix} \mathbf{k}' \\ h' \\ t' \\ j' \end{pmatrix} \partial_{\mu, \sigma \lambda}^b(\mathbf{K}) d^3 K d^3 k'. \quad (100)
 \end{aligned}$$

Introducing $\mathbf{r} - \mathbf{r}_3 = \mathbf{u}$, $\mathbf{r} - \mathbf{r}_4 = \mathbf{v}$ and polar coordinates in \mathbf{u} and \mathbf{v} , we can evaluate the integral. Furthermore, due to high concentration of the testfunctions at the origin $\mathbf{u} = 0$, $\mathbf{v} = 0$ we can replace $\exp[i\mathbf{K}(2\mathbf{r} + \mathbf{r}_3 + \mathbf{r}_4) \cdot \frac{1}{4}]$ by $\exp(i\mathbf{K}\mathbf{r})$. Thus we obtain from (100) the expression

$$I = \kappa_0 \lambda_j \int e^{i\mathbf{K}\mathbf{r}} \chi^* \begin{pmatrix} \mathbf{k} \\ h \\ t \\ j \end{pmatrix}_{\delta' \delta} \sum_{i'} \partial(\mathbf{r}, t)_{\alpha' \alpha'}^{i'} \partial_{\mu, \sigma \lambda}^b(\mathbf{r}, t) d^3 r, \quad (101)$$

where we have restored the time parameter t .

We now multiply (101) from the left with $e^{-i\mathbf{K}\mathbf{r}'} \chi(\mathbf{k}, h, t, j)_{\delta' \delta'}$ and sum over \mathbf{k}, h, t . By means of the completeness relation for superspinors we obtain in this case for (101) the result

$$\sum_{h, t} \int e^{-i\mathbf{K}\mathbf{r}'} \chi(\mathbf{k}, h, t, j)_{\delta' \delta'} I(\mathbf{k}, h, t, j) d^3 k = \kappa_0 \lambda_j \sum_{i'} \partial(\mathbf{r}', t)_{\alpha' \alpha'}^{i'} \delta_{\delta' \delta'} \delta_{\delta' \delta'} \partial_{\mu, \sigma \lambda}^b(\mathbf{r}', t) \quad (102)$$

with $I \equiv I(\mathbf{k}, h, t, j)$. Therefore (91) goes over into

$$\sum_{h, t, j} \int e^{-i\mathbf{K}\mathbf{r}'} \chi(\mathbf{k}, h, t, j)_{\delta' \delta'} (91) = \kappa_0 \lambda_j \delta_{\alpha' \alpha'} (\gamma^0 \Sigma^{\sigma \lambda} \gamma^\mu)_{\delta' \delta'} \sum_{i'} \partial(\mathbf{r}', t)_{\alpha' \alpha'}^{i'} \partial_{\mu, \sigma \lambda}^b(\mathbf{r}', t). \quad (103)$$

In the same way the other terms of (31) can be treated. So we eventually obtain

$$\frac{\partial}{\partial t} \partial(\mathbf{r}', t)_{\delta' \delta'}^j |\mathcal{G}\rangle = [\alpha_{\delta' \alpha'} \cdot \nabla' - \gamma_{\delta' \alpha'}^0 m_i] \partial(\mathbf{r}', t)_{\alpha' \alpha'}^j |\mathcal{G}\rangle - \lambda_j \kappa_0 \delta_{\alpha' \alpha'} (\gamma^0 \Sigma^{\sigma \lambda} \gamma^\mu)_{\delta' \delta'} \sum_{i'} \partial(\mathbf{r}', t)_{\alpha' \alpha'}^{i'} \partial_{\mu, \sigma \lambda}^b(\mathbf{r}', t) |\mathcal{G}\rangle. \quad (104)$$

Equation (104) is the functional formulation and description of a Dirac field coupled to anholonomic connections. The physical content of (104) can be illustrated if one retranslates the functional equation (104) into its classical precursor. This can be achieved by the ansatz, see [14],

$$|\mathcal{G}\rangle = e^{Z_0[b, j]} |0\rangle_F \quad (105)$$

with

$$Z_0[b, j] := i \int j(\mathbf{r}', t)_{\alpha' \delta'}^i \phi_{\alpha' \delta'}^i(\mathbf{r}', t) d^3 r' + \int b_{\mu, \sigma \lambda}(\mathbf{r}', t) \Gamma_{\mu, \sigma \lambda}(\mathbf{r}', t) d^3 r', \quad (106)$$

where the classical amplitudes $\phi_{\alpha' \delta'}^j(\mathbf{r}', t)$, $\Gamma_{\mu, \sigma \lambda}(\mathbf{r}', t)$ are interpreted as one-particle transition matrixelements of the corresponding quantized fields ϕ^{op} and Γ^{op} :

$$\phi_{\alpha' \delta'}^j(\mathbf{r}', t) := \langle 0 | \phi_{\alpha' \delta'}^j(\mathbf{r}', t)^{\text{op}} | \alpha \rangle, \quad \Gamma_{\mu, \sigma \lambda}(\mathbf{r}', t) := \langle 0 | \Gamma_{\mu, \sigma \lambda}(\mathbf{r}', t)^{\text{op}} | \alpha \rangle. \quad (107)$$

Substitution of (105) in (104) leads to the equation

$$\frac{\partial}{\partial t} \phi_{\sigma' \delta'}^j(\mathbf{r}', t) = [\alpha_{\sigma' \alpha'} \cdot \nabla' - i \gamma_{\sigma' \alpha'}^0 m_i] \phi_{\alpha' \delta'}^j(\mathbf{r}', t) - i \lambda_j \kappa_0 \delta_{\alpha' \alpha'} (\gamma^0 \Sigma^{\sigma \lambda} \gamma^\mu)_{\delta' \delta'} \Gamma_{\mu, \sigma \lambda}(\mathbf{r}', t) \sum_{i'} \phi_{\alpha' \delta'}^{i'}(\mathbf{r}', t), \quad (108)$$

i.e. an ordinary parity transformed Dirac equation in an external gravitational field and in complete anholonomic formulation [15]. The summation over i' and the constant λ_i are a relic of nonperturbative Pauli-Villars regularization. This relic disappears if the transition from nonobservable subfermions to observable fermions is performed. Furthermore, the transformation from anholonomic to holonomic coordinates can be performed if not only the effective fermion equation but also the effective graviton equation (29) is thoroughly investigated and its differential geometric meaning is clarified. This problem will be the subject of forthcoming papers.

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